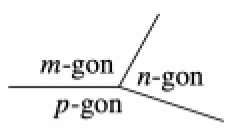
MA247. A regular m-gon, a regular n-gon and a regular p-gon share a vertex and pairwise share edges, as shown in the diagram. What is the largest possible value of p?



Originally from the 2020 Senior Mathematical Challenge (organized by the United Kingdom Mathematics Trust), Question 25.

There were 4 correct, 1 incorrect and 1 incomplete solutions submitted. We present the solution by the Missouri State University Problem Solving Group.

Suppose that $3 \leq m \leq n \leq p$. The internal angles of the three polygons are $180 - \frac{360}{m}$, $180 - \frac{360}{n}$ and $180 - \frac{360}{p}$ degrees. Since their sum is $360°$, we obtain the equation

$$\frac{1}{2} = \frac{1}{m} + \frac{1}{n} + \frac{1}{p}. \tag{1}$$

Therefore

$$\frac{1}{2} \leq \frac{3}{m},$$

whence $m \leq 6$.

When $m = 3$, (1) yields $(n-6)(p-6) = 36$. Since p is largest when n is smallest, we are led to the candidate $(m, n, p) = (3, 7, 42)$.

When $m = 4$, (1) yields $(n-4)(p-4) = 16$ and the candidate $(m, n, p) = (4, 5, 20)$.

When $m = 5$, (1) yields $(3n-10)(3p-10) = 100$ and $(m, n, p) = (5, 5, 10)$.

Finally, when $m = 6$, (1) yields $(n-3)(p-3) = 9$ and $(m, n, p) = (6, 6, 6)$.

We conclude that the largest possible value of p is 42, when a regular 42-gon meets an equilateral triangle and a regular heptagon.

Comment from the editor. All of the solutions submitted arrived at the equation $1/m + 1/n + 1/p = 1/2$. From then on, it was mainly a matter of trial and error for most of them