Problems from the American Mathematical Monthly, due August 31, 2020

12174. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.*
(a) Let n be a positive integer, and suppose that the three leading digits of the decimal expansion of 4^n are the same as the three leading digits of 5^n. Find all possibilities for these three leading digits.
(b) Prove that for any positive integer k there exists a positive integer n such that the k leading digits of the decimal expansion of 4^n are the same as the k leading digits of 5^n.

12175. *Proposed by Giuseppe Fera, Vicenza, Italy.* Let I be the incenter and G be the centroid of a triangle ABC. Prove

$$2 < \frac{IA^2}{GA^2} + \frac{IB^2}{GB^2} + \frac{IC^2}{GC^2} \leq 3.$$

12176. *Proposed by Nikolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Solve

$$xy^3 + y^2 - x^5 - 1 = 0$$

in positive integers.

12177. *Proposed by Dao Thanh Oai, Thai Binh, Vietnam, and Cherng-tiao Perng, Norfolk, VA.* Let C be a nondegenerate conic, and let l be a line. Suppose that $A_1, \ldots, A_{2n}$ and $B_1, \ldots, B_{2n}$ are points on C such that A_iA_{i+1} and B_iB_{i+1} intersect at a point on l for $i = 1, \ldots, 2n-1$.
(a) Show that $A_{2n}A_1$ and $B_{2n}B_1$ intersect at a point on l.
(b) Let $n = 3$ and take subscripts modulo 6. For $i = 1, \ldots, 6$, suppose that A_iB_i and $A_{i+1}B_{i+1}$ intersect at a point D_i. Prove that the three lines D_1D_4, D_2D_5, and D_3D_6 are concurrent.

12178. *Proposed by Stephen Portnoy, University of Illinois, Urbana, IL.* Given any function $f\colon \mathbb{R} \to \mathbb{R}$, show that there is a real number x and a sequence $x_1, x_2, \ldots$ of distinct real numbers such that $x_n \to x$ and $f(x_n) \to f(x)$ as $n \to \infty$.

12179. *Proposed by Nick MacKinnon, Winchester College, Winchester, UK.* A positive integer n is *good* if its prime factorization $2^{a_1}3^{a_2}\cdots p_m^{a_m}$ has the property that a_i/a_{i+1} is an integer whenever $1 \leq i < m$. Find all n greater than 2 such that $n!$ is good.

12180. *Proposed by Pablo Fernández Refolio, Madrid, Spain.* Prove

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}^2}{2^{8n}(2n+1)} = \frac{2}{\pi} - \frac{\sqrt{2}\,C^2}{\pi^{3/2}} + \frac{\sqrt{2\pi}}{2C^2},$$

where $C = \int_0^\infty t^{-1/4}e^{-t}\,dt$.

Problems from Mathematics Magazine, due September 1, 2020

2091. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bârlad, Romania.*

Let ABC be a triangle with sides of lengths a, b, c, altitudes h_a, h_b, h_c, inradius r, and circumradius R. Prove that the following inequality holds:

$$h_a + h_b + h_c \geq 9r + \frac{a^2 + b^2 + c^2 - ab - ac - bc}{4R},$$

with equality if and only if $\triangle ABC$ is equilateral.

2092. *Proposed by Seán M. Stewart, Bomaderry, Australia.*

Let n be a non-negative integer. Evaluate

$$\int_0^\infty \frac{1}{x^{2n+3}} \left(\sin x - \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right) dx.$$

2093. *Proposed by Jacob Siehler, Gustavus Adolphus College, Saint Peter, MN.*

Suppose π is a permutation of $\{1, 2, \ldots, 2m\}$, where m is a positive integer. Consider the (possibly empty) subsequence of $\pi(m+1), \pi(m+2), \ldots, \pi(2m)$ consisting of only those values which exceed $\max\{\pi(1), \ldots, \pi(m)\}$. Let $P(m)$ denote the probability that this subsequence never decreases (note that the empty sequence has this property), when π is a randomly chosen permutation of $\{1, \ldots, 2m\}$. Evaluate $\lim_{m\to\infty} P(m)$.

2094. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Find the smallest number $f(n)$ such that for any set of unit vectors $x_1, \ldots, x_n$ in $\mathbb{R}^n$, there is a choice of $a_i \in \{-1, 1\}$ such that $|a_1 x_1 + \cdots + a_n x_n| \leq f(n)$.

2095. *Proposed by Mircea Merca, University of Craiova, Romania.*

Show that

$$\sum_{k=1}^{n} k \left\lfloor \frac{n+1-k}{d} \right\rfloor = \begin{cases} \lceil (n+1)(n-1)(2n+3)/24 \rceil & \text{if } d = 2 \\ \lceil (n+1)^2(n-2)/18 \rceil & \text{if } d = 3 \\ \lceil (n+1)(2n+1)(n-3)/48 \rceil & \text{if } d = 4 \\ \lceil (n+1)n(n-4)/30 \rceil & \text{if } d = 5 \end{cases}$$

Problems from Crux MathemAttic, due September 15, 2020

MA76. The sum of two real numbers is n and the sum of their squares is $n+19$, for some positive integer n. What is the maximum possible value of n?

MA77. In a regular decagon, all diagonals are drawn. If a diagonal is chosen at random, what is the probability that it is neither one of the shortest nor one of the longest?

MA78. Let $T(n)$ be the digit sum of a positive integer n; for example, $T(5081) = 5+0+8+1 = 14$. Find the number of three-digit numbers that satisfy $T(n) + 3n = 2020$.

MA79. Suppose BD bisects $\angle ABC$, $BD = 3\sqrt{5}$, $AB = 8$ and $DC = \frac{3}{2}$. Find $AD + BC$.

MA80. Suppose $ABCD$ is a parallelogram. Let E and F be two points on BC and CD, respectively. If $CE = 3BE$, $CF = DF$, DE intersects AF at K and $KF = 6$, find AK.

Problems from Crux Olympiad Corner, due September 15, 2020

OC486. There are 2017 points in the plane such that among any three of them two can be selected so that their distance is less than 1. Prove that there is a circle of radius 1 containing at least 1009 of the given points.

OC487. Let a, b, c be real numbers such that $1 < b \leq c^2 \leq a^{10}$ and

$$\log_a b + 2\log_b c + 5\log_c a = 12.$$

Show that

$$2\log_a c + 5\log_c b + 10\log_b a \geq 21.$$

OC488. Prove that the equation

$$(x^2 + 2y^2)^2 - 2(z^2 + 2t^2)^2 = 1$$

has infinitely many integer solutions.

OC489. The incircle of a triangle ABC touches AB and AC at points D and E, respectively. Point J is the center of the excircle of triangle ABC tangent to side BC. Points M and N are midpoints of segments JD and JE, respectively. Lines BM and CN intersect at point P. Prove that P lies on the circumcircle of triangle ABC.

OC490. Find the smallest prime number that cannot be written in the form $|2^a - 3^b|$ with nonnegative integers a, b.

Problems from Crux Mathematicorum, due September 15, 2020

4551. *Proposed by Michel Bataille.*

Let ABC be a triangle with sides $BC = a, CA = b$ and $AB = c$. Suppose $b > c$ and let A_1, A_2 be the two points such that ΔA_1BC and ΔA_2BC are equilateral. Express the circumradius of ΔAA_1A_2 as a function of a, b, c.

4552. *Proposed by Anupam Datta.*

Given positive integers a, b and n, prove that the following are equivalent:

1. $b \equiv ax \pmod{n}$ has a solution with $\gcd(x, n) = 1$;
2. $b \equiv ax \pmod{n}$ and $a \equiv by \pmod{n}$ have solutions $x, y \in \mathbb{Z}$;
3. $\gcd(a, n) = \gcd(b, n)$.

4553. *Proposed by Daniel Sitaru.*

Find

$$\lim_{n\to\infty}\left(\frac{\int_0^1 x^2(x+n)^n dx}{(n+1)^n}\right)$$

4554. *Proposed by George Stoica.*

Let ϵ be a given constant with $0 < \epsilon < 1$, and let (a_n) be a sequence with $0 \le a_n < \epsilon$ for all $n \ge 1$. Prove that $(1 - a_n)^n \to 1$ as $n \to \infty$ if and only if $na_n \to 0$ as $n \to \infty$.

4555. *Proposed by Michael Rozenberg and Leonard Giugiuc.*

Prove that if a, b, c and d are positive numbers that satisfy

$$ab + bc + cd + da + ac + bd = 6,$$

then

$$a + b + c + d \ge 2\sqrt{(a^2 + b^2 + c^2 + d^2)abcd}.$$

When does the equality hold?

4556. *Proposed by Marian Cucoanes and Lorean Saceanu.*

Let $x \ge 1$ be a real number and consider a triangle ABC. Prove that

$$\frac{(x-\cos A)(x-\cos B)(x-\cos C)}{(x+\cos A)(x+\cos B)(x+\cos C)} \le \left[\frac{(3x-1)R-r}{(3x+1)R+r}\right]^3.$$

When does the equality hold?

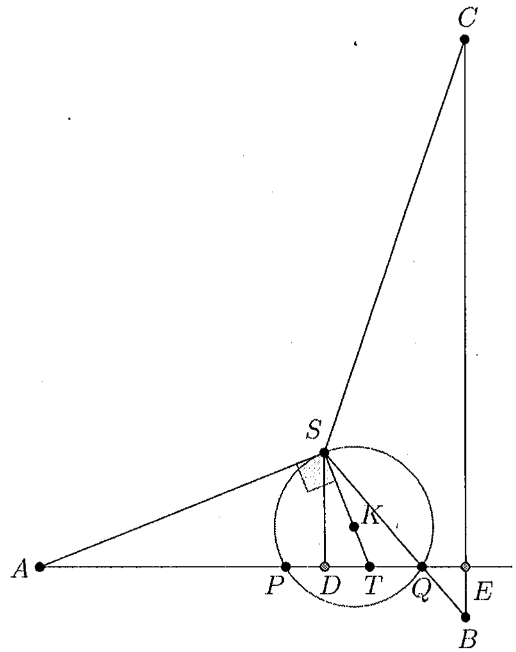
4557. *Proposed by George Apostolopoulos.*

Let m_a, m_b and m_c be the lengths of the medians of a triangle ABC with circumradius R and inradius r. Let a, b and c be the lengths of the sides of ABC. Prove that

$$\frac{24r^2}{R} \le \frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \le \frac{4r^2 - 2Rr}{r}.$$

4558. *Proposed by Thanos Kalogerakis, Leonard Giugiuc and Kadir Altintas.*

Consider a diagram below, where triangle SAT is right-angled and $\tan T > 2$. The point K lies on the segment ST such that $SK = 2KT$. The circle centered at K with radius KS intersects the line AT at P and Q. Point D is the projection of S on AT and E is a point on AT such that D lies on AE and $AD = 2DE$. Finally, suppose that SQ and SP intersect the perpendicular at E on AT at B and C respectively. Prove that S is the incenter of the triangle ABC.



4559. *Proposed by Nho Nguyen Van.*

Let x_k be positive real numbers. Prove that for every natural number $n \geq 2$, we have

$$\left(\sum_{k=1}^{n} x_k^{10}\right)^3 \geq \left(\sum_{k=1}^{n} x_k^{15}\right)^2$$

4560. *Proposed by Mihaela Berindeanu.*

Let E and F be midpoints on the respective sides CA and AB of triangle ABC, and let P be the second point of intersection of the circles ABE and ACF. Prove that the circle AEF intersects the line AP again in the point X for which $AX = 2XP$.

Problems from the College Mathematics Journal, due September 15, 2020

1171. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, and c be the roots of the equation $x^3 - 2x^2 - x + 1 = 0$, with $a < b < c$. Find the value of the expression $(\frac{a}{b})^2 + (\frac{b}{c})^2 + (\frac{c}{a})^2$.

1172. *Proposed by Xiang-Qian Chang, MCPHS University, Boston, MA.*

Suppose that a function $y = y(x)$ satisfies the following first-order differential equation:

$$y' + x^6 - x^4 - 2yx^3 - 3x^2 + yx + y^2 - 1 = 0,$$

with initial value $y(0) = \sqrt{\frac{\pi}{2}}$. Show that $y(x) \sim \frac{1+x^4}{x}$ as x tends to infinity.

1173. *Proposed by Greg Oman, University of Colorado, Colorado Springs, Colorado Springs, CO.*

All rings in this problem are assumed commutative with identity. Now, let R and S be rings and suppose that R is a subring of S (we assume that the identity of R is the identity of S). An element $s \in S$ is *integral over* R if s is a root of a monic polynomial $f(x) \in R[x]$. If we set $\overline{R} := \{s \in S : s \text{ is integral over } R\}$, then it is well-known that $\overline{R}$ is a subring of S containing R. The ring $\overline{R}$ is called the *integral closure of R in S*. In case $\overline{R} = S$, then we say that *S is integral over R*. For a ring R, let $R^\times$ denote the multiplicative group of units of R. Prove or disprove: for every infinite integral domain D_1, there exists an integral domain D_2 such that D_2 is integral over D_1 and $|D_2^\times| = |D_1|$ (i.e., the set of units of D_2 has the same cardinality as that of D_1).

1174. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be complex numbers which are not integers. Prove that the infinite product below converges if and only if $\sum_{i=1}^k a_i = \sum_{i=1}^k b_i$. What is the value of the product?

$$\prod_{n=1}^{\infty} \frac{(n - a_1)(n - a_2) \cdots (n - a_k)}{(n - b_1)(n - b_2) \cdots (n - b_k)}$$

1175. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let F_1 and F_2 be distinct proper subfields of the field $\mathbb{R}$ of real numbers. Is there a field isomorphism $f : F_1 \to F_2$ preserving signs, that is, for all real x: $x \in F_1$ and $x > 0$ if and only if $f(x) \in F_2$, $f(x) > 0$?

Problems from the American Mathematical Monthly, due September 30, 2020

12181. *Proposed by Shivam Sharma, University of Delhi, New Delhi, India.* Prove

$$\sum_{k=2}^{\infty} \frac{1}{k} \int_0^1 \left\{ \frac{1}{\sqrt[k]{x}} \right\} dx = \gamma,$$

where $\{x\}$ equals $x - \lfloor x \rfloor$, the fractional part of x, and γ is $\lim_{n\to\infty}\left(-\ln n + \sum_{i=1}^{n}(1/i)\right)$, the Euler–Mascheroni constant.

12182. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let R and r be the circumradius and inradius, respectively, of triangle ABC. Let D, E, and F be chosen on sides BC, CA, and AB so that AD, BE, and CF bisect the angles of ABC. Let R_A, R_B, and R_C denote the circumradii of triangles AEF, BFD, and CDE, respectively. Prove $R_A + R_B + R_C \le 3R^2/(4r)$.

12183. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let $\left[{n \atop k}\right]_q$ denote the Gaussian binomial coefficient

$$\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}.$$

For integers m, n, and r with $m \ge 1$ and $n \ge r \ge 0$, prove

$$\sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}-rk}}{1-q^{k+m}} \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{q^{rm}}{1-q^m} \begin{bmatrix} m+n \\ m \end{bmatrix}_q^{-1}.$$

12184. *Proposed by Paolo Perfetti, Universitá degli Studi di Roma "Tor Vergata," Rome, Italy.* Prove

$$\int_1^{\infty} \frac{\ln(x^4 - 2x^2 + 2)}{x\sqrt{x^2-1}}\, dx = \pi \ln(2+\sqrt{2}).$$

12185. *Proposed by George Stoica, Saint John, NB, Canada.* Let $n_1, \ldots, n_k$ be pairwise relatively prime integers greater than 1. For $i \in \{1, \ldots k\}$, let $f_i(x) = \sum_{m=1}^{n_i} x^{m-1}$. Let A be a $2n$-by-$2n$ matrix with real entries such that $\det f_j(A) = 0$ for all $j \in \{1, \ldots, k\}$. Prove $\det A = 1$.

12186. *Proposed by Anatoly Eydelzon, University of Texas at Dallas, Richardson, TX.* For $v = \langle x_1, \ldots, x_n\rangle$ in $\mathbb{R}^n$, let $\|v\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ and $\|v\|_\infty = \max_{1\le i\le n} |x_i|$; these are the usual p-norm and ∞-norm on $\mathbb{R}^n$. For what v does the series

$$\sum_{p=1}^{\infty} \left(\|v\|_p - \|v\|_\infty \right)$$

converge?

12187. *Proposed by Khakimboy Egamberganov, Sorbonne University, Paris, France.* Given a scalene triangle ABC, let M be the midpoint of BC, and let m and s denote the median and symmedian lines, respectively, from A. (The *symmedian* line from A is the reflection of the median from A across the angle bisector from A.) Let K be the projection of C onto m, and let L be the projection of B onto s. Let P be the intersection of BL and CK, and let Q be the intersection of KL and BC. Prove that PM and AQ are perpendicular.